Wave propagation in Lossy Media

wave equation: $\nabla^2 \tilde{E} - \gamma^2 \tilde{E} = 0$

$$\gamma^2 = -\omega^2 \mu \epsilon_c = -\omega^2 \mu (\epsilon' - j\epsilon'')$$

$\epsilon' = \epsilon$

$\epsilon'' = \frac{\sigma}{\omega}$

$$\gamma = \alpha + j\beta$$

$$(\alpha + j\beta)^2 = (\alpha^2 - \beta^2) - j2\alpha\beta = -\omega^2 \mu \epsilon' + j\omega^2 \mu \epsilon''$$

solve for α and β:

$$\alpha = \omega \left[\frac{\mu\epsilon'}{2} \left[\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'}\right)^2} - 1 \right] \right]^{1/2}$$

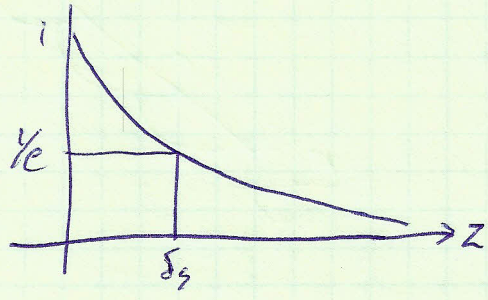
$$\beta = \omega \left[\frac{\mu\epsilon'}{2} \left[\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'}\right)^2} + 1 \right] \right]^{1/2}$$

$$\tilde{E}(z) = \hat{x}\tilde{E}_x(z) = \hat{x}E_{x0}e^{-\gamma z} = \hat{x}E_{x0}e^{-\alpha z}e^{-j\beta z}$$

get H by $\nabla \times \tilde{E} = -j\omega\mu\tilde{H}$ or $\tilde{H} = (\hat{k} \times \tilde{E})/\eta_c$

where $\eta_c = \sqrt{\frac{\mu}{\epsilon_c}} = \sqrt{\frac{\mu}{\epsilon_1}}\left(1 - j\frac{\epsilon''}{\epsilon'}\right)^{-1/2}$

Skin depth $\delta_s = \frac{1}{\alpha}$



- Low-Loss Dielectric

$$\gamma = j\omega\sqrt{\mu\epsilon'}\left(1 - j\frac{\epsilon''}{\epsilon'}\right)^{1/2}$$

for $|x| \ll 1$, $(1-x)^{1/2} \approx 1 - \frac{x}{2}$

$$\gamma \approx j\omega\sqrt{\mu\epsilon'}\left(1 - j\frac{\epsilon''}{2\epsilon'}\right)$$

$$\beta \approx \omega\sqrt{\mu\epsilon'} = \omega\sqrt{\mu\epsilon}$$

$$\alpha \approx \frac{\omega\epsilon''}{2}\sqrt{\frac{\mu}{\epsilon'}} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} \longleftarrow \text{use } \epsilon'' = \frac{\sigma}{\omega}$$

note that β is nearly identical to lossless case

$$\eta_c \approx \sqrt{\frac{\mu}{\epsilon'}}\left(1 + j\frac{\epsilon''}{2\epsilon'}\right) \approx \sqrt{\frac{\mu}{\epsilon}}$$

- Good Conductor

$$\frac{\epsilon''}{\epsilon'} \gg 100$$

$$\alpha \approx \omega\sqrt{\frac{\mu\epsilon''}{2}} = \omega\sqrt{\frac{\mu\sigma}{2\omega}} = \sqrt{\pi f\mu\sigma} \qquad \left(\text{and } \delta_s = \frac{1}{\alpha}\right)$$

$$\beta \approx \alpha$$

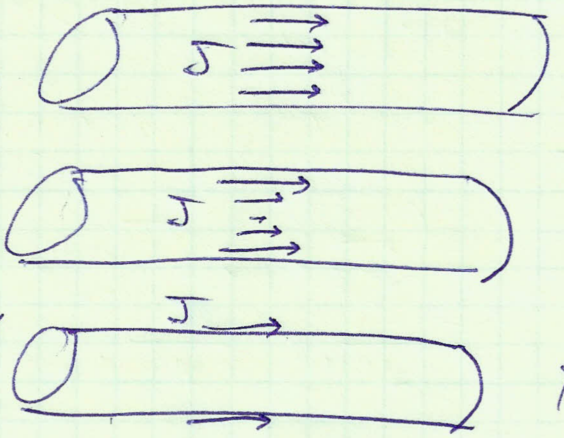
use $\sqrt{j} = \frac{1+j}{\sqrt{2}}$

$$\eta_c \approx \sqrt{j\frac{\mu}{\epsilon''}} = (1+j)\sqrt{\frac{\pi f\mu}{\sigma}} = (1+j)\frac{\alpha}{\sigma}$$

for perfect conductor : $\alpha = \beta = 0$, $\eta_c = 0$

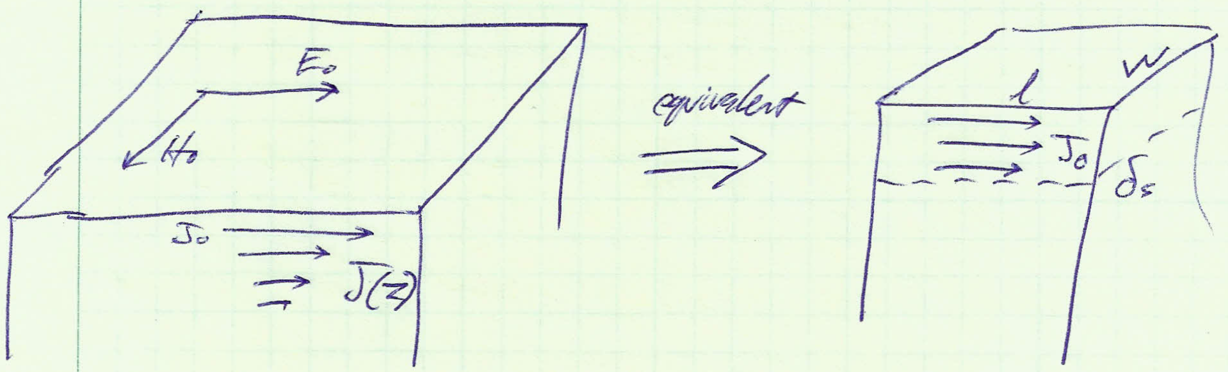
for real conductor, impedance has inductive component equal to resistive part!

Current flow in a good conductor

uniform  DC case

e^{-x/δ_s} AC case

infinitesimal surface current Perfect conductor case



$$E(z) = \hat{x} E_o e^{-\alpha z} e^{-j\beta z}$$

for good conductor, $\alpha = \beta = 1/\delta_s$

$$\tilde{J}_x(z) = \sigma E_o e^{-\alpha z} e^{-j\beta z} = J_o e^{-\alpha z} e^{-j\beta z}$$

$$\tilde{J}_x(z) = J_o e^{-(1+j) z/\delta_s}$$

$$I = w\int_0^\infty \tilde{J}_x(z)\,dz = w\int_0^\infty J_o e^{-(1+j)z/\delta_s}\,dz = \frac{J_o w \delta_s}{(1+j)}$$

Voltage across length l is $E_o l = \frac{J_o}{\sigma} l$

$$Z = \frac{\tilde{V}}{I} = \frac{1+j}{\sigma\delta_s}\cdot\frac{l}{w} \quad \text{or} \quad Z = Z_s \frac{l}{w}$$

$$Z_s = \frac{1+j}{\sigma\delta_s} = \text{surface impedance}$$

$Z_s = R_s + j\omega L_s$

$R_s = \frac{1}{\sigma\delta_s} = \sqrt{\frac{\pi f \mu}{\sigma}}$ resistance equivalent to slab of thickness δ_s with conductivity σ

$L_s = \frac{1}{\omega\sigma\delta_s} = \frac{1}{2}\sqrt{\frac{\mu}{\pi f \sigma}}$

- Electromagnetic Power Density

Poynting Vector $\vec{S} = \vec{E} \times \vec{H}$ (W/m²)

$P = \int_A \vec{S} \cdot \hat{n}\, dA$ total power through surface A

$\vec{S}_{av} = \frac{1}{2}\mathrm{Re}\left[\tilde{E} \times \tilde{H}^*\right]$ average power density

similar to $P(z,t) = v(z,t)\,i(z,t)$

similar to

$P_{av}(z) = \frac{1}{2}\mathrm{Re}\left[\tilde{V}(z)\tilde{I}^*\right]$

- In a lossless medium:

$|\tilde{E}| = (\tilde{E} \cdot \tilde{E}^*)^{1/2}$

$\tilde{H} = \frac{1}{\eta}\hat{k} \times \tilde{E}$

$\vec{S}_{av} = \hat{k}\frac{|\tilde{E}|^2}{2\eta}$

- In a lossy medium:

$\tilde{E}(z) = E_0 e^{-\alpha z} e^{-j\beta z}$, $\tilde{H}(z) = \frac{1}{\eta_c}\hat{k} \times E_0 e^{-\alpha z} e^{-j\beta z}$

$\vec{S}_{av}(z) = \frac{1}{2}\mathrm{Re}\left[\tilde{E} \times \tilde{H}^*\right] = \hat{k}\frac{|E_0|^2}{2}e^{-2\alpha z}\,\mathrm{Re}\left(\frac{1}{\eta_c^*}\right)$

note E and H decay as $e^{-\alpha z}$, but S decays as $e^{-2\alpha z}$